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# ON FINITE DEFLECTIONS OF ANISOTROPIC LAMINATED ELASTIC PLATES†

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**Abstract**—The purposes of this paper are to (1) deduce a generalization of Von Karman's analysis, incorporating the effects of transverse shear deformations and of laminations, with the planes of the laminations not necessarily parallel to the plane of the plate; (2) further pursue the possibility of deriving plate equations from three-dimensional elasticity through use of a variational equation for displacements and transverse stresses; (3) consider the possibility of utilizing finite-elasticity variational formulations, including independent translational and rotational displacement components for the derivation of approximate two-dimensional results from three-dimensional theory.

## INTRODUCTION

The considerations which follow have three different purposes. One purpose is to deduce a generalization of von Karman's analysis of plates, incorporating the effects of transverse shear deformation and of laminations, with the planes of the laminations not necessarily parallel to the midplane of the plate. A second purpose is the further pursuit of a possibility of deriving two-dimensional plate equations from three-dimensional elasticity through the use of a mixed variational equation for displacements and *transverse* stresses, in contrast to corresponding considerations involving displacements and *all* stresses, or displacements alone, or stresses alone. The third purpose is to consider the possibility of utilizing finite-elasticity variational formulations involving rotational displacements in addition to translational displacements for direct-methods derivations of approximate two-dimensional results from three-dimensional theory.

As regards the existence of related work we limit ourselves here to making reference to an analysis of the problem of *infinitesimal* deflections of laminated plates using a variational theorem for translational displacements and *transverse* stresses[3], and to a *formulation* of a variational theorem in finite elasticity for translational and rotational displacements and transverse stresses[2]. Other previous work on the derivation of linear two-dimensional plate equations by direct methods variational procedures is too well known to require enumeration in this place. As regards the relevant literature on variational theorems in finite elasticity, including in particular formulations by Frayes de Veubeke, Wempner, Bufler and Atluri, we here refer the reader to the list of references in Refs [1, 2].

## THE VARIATIONAL EQUATION

Given a system of Cartesian coordinates $x_i$ we consider an elastic layer $-c \leqslant x_3 \leqslant c$, bounded by a cylindrical surface $f(x_1, x_2) = 0$. We assume, for ease of discussion of the essential aspects of what follows, that the faces $x_3 = \pm c$ of the layer are traction free and we do not here concern ourselves with the *details* of the traction distribution over the cylindrical boundary portion. Given this *three*-dimensional problem we develop our analysis to the point of establishing a system of *two*-dimensional differential equations, without the simultaneous derivation of a consistent system of two-dimensional boundary conditions, which we know to be possible.

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We take as our point of departure a variational equation of the form

$$\delta\int\int\int[V(\gamma_{11},\hat{\gamma}_{12},\gamma_{22};\hat{\tau}_{13},\hat{\tau}_{23},\tau_{33})+\frac{1}{2}(\tau_{12}-\tau_{21})(\gamma_{12}-\gamma_{21})$$
$$+\gamma_{13}\tau_{13}+\gamma_{31}\tau_{31}+\gamma_{23}\tau_{23}+\gamma_{32}\tau_{32}+\gamma_{33}\tau_{33}]\,dx_3\,dx_1\,dx_2=0, \quad (1)$$

where

$$\hat{\gamma}_{12}=\gamma_{12}+\gamma_{21},\qquad \hat{\tau}_{13}=\frac{1}{2}(\tau_{13}+\tau_{31}),\qquad \hat{\tau}_{23}=\frac{1}{2}(\tau_{23}+\tau_{32}). \quad (2)$$

In this the quantities $\tau_{ij}$ are components of stress which we have, in the present context[1, 2], designated as distinguished generalized Piola components. The quantities $\gamma_{ij}$ are components of strain, which are *conjugate* to the $\tau_{ij}$, inasmuch as the *virtual work* of the stresses $\tau_{ij}$ is of the form $\tau_{ij}\delta\gamma_{ij}$.

Expressions for the $\gamma_{ij}$, which we have earlier called generalized displacement gradient components[1, 2], can be written in the form

$$\gamma_{ij}=(\delta_{ik}+u_{k,i})\alpha_{jk}-\delta_{ij}, \quad (3)$$

in terms of the components of a displacement vector $\mathbf{u}=u_k\mathbf{e}_k$, and in terms of the components of an orthogonal unit vector triad $\mathbf{t}_j=\alpha_{jk}\mathbf{e}_k$ which is used in the stress vector representation $\boldsymbol{\tau}_i=\tau_{ij}\mathbf{t}_j$.

With eqns (3) and the *partial* set of constitutive relations

$$\tau_{11}=\partial V/\partial\gamma_{11},\qquad \tau_{22}=\partial V/\partial\gamma_{22},\qquad \hat{\tau}_{12}=\partial V/\partial\hat{\gamma}_{12} \quad (4)$$

as constraint equations for the variational equation (1), the Euler equations of (1) come out to be the complementary set of constitutive relations, in the form

$$\hat{\gamma}_{13}=-\partial V/\partial\hat{\tau}_{13},\qquad \hat{\gamma}_{23}=-\partial V/\partial\hat{\tau}_{23},\qquad \gamma_{33}=-\partial V/\partial\tau_{33}, \quad (5)$$

together with equations of force and moment equilibrium of the form

$$(\alpha_{jk}\tau_{ij})_{,i}=0,\qquad (\delta_{in}+u_{n,i})e_{mjk}\alpha_{mn}\tau_{ij}=0. \quad (6a, b)$$

In order to derive (5) and (6a, b) we use the fact that the variations $\delta\tau_{i3}$, $\delta\tau_{3i}$, $\delta(\tau_{12}-\tau_{21})$ and $\delta u_k$ are independent, and the the variations $\delta\alpha_{jk}$ are, as a consequence of the orthogonality equations $\alpha_{ik}\alpha_{jk}=\delta_{ij}$, expressible in the form $\alpha_{ik}\delta\alpha_{jk}=e_{ijm}\delta\chi_m$, in terms of three independent variations $\delta\chi_m$.

If we limit ourselves, as we shall do in what follows, to the range of *small* finite rotations, in terms of rotational parameters $\beta_i$, in such a way as to retain no more than first degree terms in $\beta_3$ and second degree terms in $\beta_1$ and $\beta_2$ then the quantities $\alpha_{jk}$ in (3) come out to be, in accordance with the analysis in Appendix B,

$$\alpha_{11}=1-\frac{1}{2}\beta_1^2,\qquad \alpha_{12}=\beta_3-\frac{1}{2}\beta_1\beta_2,\qquad \alpha_{13}=\beta_1,$$
$$\alpha_{21}=-\beta_3-\frac{1}{2}\beta_1\beta_2,\qquad \alpha_{22}=1-\frac{1}{2}\beta_2^2,\qquad \alpha_{23}=\beta_2, \quad (7)$$
$$\alpha_{31}=-\beta_1,\qquad \alpha_{32}=-\beta_2,\qquad \alpha_{33}=1-\frac{1}{2}\beta_1^2-\frac{1}{2}\beta_2^2,$$

and, if we further assume *small* finite strains, the quantities $\gamma_{ij}$ come out to be

$$\gamma_{11} = u_{1,1}+u_{3,1}\beta_1-\frac{1}{2}\beta_1^2, \qquad \gamma_{12} = u_{2,1}-\beta_3+u_{3,1}\beta_2-\frac{1}{2}\beta_1\beta_2,$$
$$\gamma_{22} = u_{2,2}+u_{3,2}\beta_2-\frac{1}{2}\beta_2^2, \qquad \gamma_{21} = u_{1,2}+\beta_3+u_{3,2}\beta_1-\frac{1}{2}\beta_1\beta_2, \tag{8}$$

and

$$\gamma_{13} = u_{3,1}-\beta_1, \qquad \gamma_{31} = u_{1,3}+{}_1,$$
$$\gamma_{23} = u_{3,2}-\beta_2, \qquad \gamma_{32} = u_{2,3}+\beta_2, \tag{9}$$

$$\gamma_{33} = u_{3,3}-u_{1,3}\beta_1-u_{2,3}\beta_2-\frac{1}{2}\beta_1^2-\frac{1}{2}\beta_2^2, \tag{10}$$

with independent variations $\delta\beta_i$ in place of the variations $\delta\chi_m$.

For the derivation of an approximate two-dimensional plate theory we will make use of eqn (1) in the developed form

$$\iiint\Bigg[\frac{\partial V}{\partial\gamma_{11}}\delta\gamma_{11}+\frac{\partial V}{\partial\gamma_{22}}\delta\gamma_{22}+\left(\frac{\partial V}{\partial\hat{\gamma}_{12}}+\frac{\tau_{12}-\tau_{21}}{2}\right)\delta\gamma_{12}+\left(\frac{\partial V}{\partial\hat{\gamma}_{12}}-\frac{\tau_{12}-\tau_{21}}{2}\right)\delta\gamma_{21}$$
$$+(\gamma_{12}-\gamma_{21})\delta\frac{\tau_{12}-\tau_{21}}{2}+\left(\frac{1}{2}\frac{\partial V}{\partial\hat{\tau}_{i3}}+\gamma_{i3}\right)\delta\tau_{i3}+\left(\frac{1}{2}\frac{\partial V}{\partial\hat{\tau}_{i3}}+\gamma_{3i}\right)\delta\tau_{3i}$$
$$+\left(\frac{\partial V}{\partial\tau_{33}}+\gamma_{33}\right)\delta\tau_{33}+\tau_{i3}\delta\gamma_{i3}+\tau_{3i}\delta\gamma_{3i}+\tau_{33}\delta\gamma_{33}\Bigg]\,dx_3\,dx_1\,dx_2 = 0. \tag{11}$$

## DERIVATION OF APPROXIMATE TWO-DIMENSIONAL THEORY

Given various known results concerning two-dimensional theories of sixth or higher order for shear-deformable plates, through use of the variational theorems for stresses or displacements for the case of infinitesimal deformations, and through use of the variational theorem for displacements in terms of Green strains for small finite deformations, we here concern ourselves in particular with two specific questions. The first of these concerns the effect of the occurrence of the rotational displacement variables $\beta_i$, alongside the translational displacement variables $u_i$, in the mixed variational theorem for finite deformations in eqn (1). Eventually, we hope to arrive at a conclusion that use of the $\beta_i$ in conjunction with the $u_i$, in place of the $u_i$ alone with the conventional variational theorem in terms of Green strains, brings with it clearly identifiable advantages. For the present we limit ourselves to placing on record the simplest such result which we have been able to obtain. The second question concerns the form in which the effect of the transverse shearing stresses will appear in the system of two-dimensional constitutive equations if we allow for such anisotropy in a laminated material as is possible when the faces of the laminations are not necessarily parallel to the faces $x_3 = \pm c$ of the plate.

We begin our derivation by stipulating the well-known translational displacement component approximations

$$(u_1, u_2) = (v_1, v_2)+(\psi_1, \psi_2)x_3, \qquad u_3 = w, \tag{12}$$

with $v_i$, $\psi_i$ and $w$ being functions of $x_1$ and $x_2$.

The second step in our derivation is the nonconventional introduction of analogous approximations for the rotational variables $\beta_i$. Given the appearance of the strain components in eqns (8) and (9), and given the nature of the approximations in (12) the simplest appropriate choice of the $\beta_i$ is thought to be

$$(\beta_1, \beta_2) = (\phi_1, \phi_2), \qquad \beta_3 = \omega + \phi_3 x_3, \tag{13}$$

with $\phi_i$ and $\omega$ being functions of $x_1$ and $x_2$.

The third step consists in choosing approximations for the stress components $\tau_{i3}$ and $\tau_{3i}$, and for the reactive stress quantity $\tau_{12}-\tau_{21}$. In this account we are limiting ourselves to the simplest possible rational approximations for these components by setting

$$(\tau_{i3}, \tau_{3i}) = (Q_{i3}, Q_{3i})/2c, \qquad i = 1, 2, \tag{14}$$

$$\tau_{33} = 0, \qquad \tau_{12}-\tau_{21} = 0. \tag{15}$$

In writing (14) and (15) we have avoided making the further approximations which would consist in stipulating $Q_{i3} = Q_{3i}$, and we note that for problems involving distributed transverse surface loads we should replace the approximation $\tau_{33} = 0$ by an expression accounting for these surface loads. We note further that with the assumed general character of anisotropy and nonhomogeneity in thickness direction, and with the way in which the stresses $\tau_{i3}$ and $\tau_{3i}$ appear in the variational equation (11), no other one-term approximations for these stresses would *generally* be superior to the ones made in (14).

Having now (12) to (15) we take account of the fact that the assumptions (15) reduce the variational equation (11) to the abbreviated form

$$\iiint \left[ \frac{\partial V}{\partial \gamma_{11}} \delta\gamma_{11} + \frac{\partial V}{\partial \gamma_{22}} \delta\gamma_{22} + \frac{\partial V}{\partial \hat{\gamma}_{12}} \delta\hat{\gamma}_{12} + \left( \frac{1}{2} \frac{\partial V}{\partial \hat{\tau}_{13}} + \gamma_{i3} \right) \delta\tau_{i3} + \left( \frac{1}{2} \frac{\partial V}{\partial \hat{\tau}_{i3}} + \gamma_{3i} \right) \delta\tau_{3i} + \tau_{i3}\delta\gamma_{i3} + \tau_{3i}\delta\gamma_{3i} \right] \mathrm{d}x_3\, \mathrm{d}x_1\, \mathrm{d}x_2 = 0, \tag{16}$$

and that, as a consequence, the strain component $\gamma_{33}$ in (10) disappears from our consideration, while the strain components $\gamma_{12}$ and $\gamma_{21}$ occur only as the combination $\hat{\gamma}_{12}$. With this, and with (12) and (13), we now write

$$(\gamma_{11}, \gamma_{22}, \hat{\gamma}_{12}) = (\varepsilon_{11}, \varepsilon_{22}, \hat{\varepsilon}_{12}) + (\kappa_{11}, \kappa_{22}, \hat{\kappa}_{12}) x_3, \tag{17}$$

where

$$\varepsilon_{11} = v_{1,1} + w_{,1}\phi_1 - \frac{1}{2}\phi_1^2, \qquad \varepsilon_{22} = v_{2,2} + w_{,2}\phi_2 - \frac{1}{2}\phi_2^2,$$

$$\hat{\varepsilon}_{12} = v_{1,2} + v_{2,1} + w_{,2}\phi_1 + w_{,1}\phi_2 - \phi_1\phi_2, \tag{18}$$

$$\kappa_{11} = \psi_{1,1}, \qquad \kappa_{22} = \psi_{2,2}, \qquad \hat{\kappa}_{12} = \psi_{1,2} + \psi_{2,1}, \tag{19}$$

$$\gamma_{i3} = w_{,i} - \phi_i, \qquad \gamma_{3i} = \psi_i + \phi_i. \tag{20}$$

The introduction of (17) and (14), in conjunction with an observation of the $x_3$-independence of $\gamma_{i3}$ and $\gamma_{3i}$ in (20), and in conjunction with the defining relations

$$(N_{11}, N_{22}, N_{12}) = \int_{-c}^{c} \left( \frac{\partial V}{\partial \gamma_{11}}, \frac{\partial V}{\partial \gamma_{22}}, \frac{\partial V}{\partial \hat{\gamma}_{12}} \right) \mathrm{d}x_3, \tag{21}$$

$$(M_{11}, M_{22}, M_{12}) = \int_{-c}^{c} \left( \frac{\partial V}{\partial \gamma_{11}}, \frac{\partial V}{\partial \gamma_{22}}, \frac{\partial V}{\partial \hat{\gamma}_{12}} \right) x_3\, \mathrm{d}x_3, \tag{22}$$

$$\gamma_i = -\frac{1}{2c}\int_{-c}^{c}\frac{\partial V}{\partial \hat{\tau}_{i3}}\,\mathrm{d}x_3, \tag{23}$$

reduces the triple integral relation (16) to a double integral relation, of the form

$$\iint\Big[N_{11}\delta\varepsilon_{11}+N_{12}\delta\hat{\varepsilon}_{12}+N_{22}\delta\varepsilon_{22}+M_{11}\delta\kappa_{11}+M_{12}\delta\hat{\kappa}_{12}+M_{22}\delta\kappa_{22}+Q_{i3}\delta\gamma_{i3}$$
$$+Q_{3i}\delta\gamma_{3i}+\left(\gamma_{3i}-\frac{1}{2}\gamma_i\right)\delta Q_{3i}+\left(\gamma_{i3}-\frac{1}{2}\gamma_i\right)\delta Q_{i3}\Big]\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0. \tag{24}$$

With $\delta\varepsilon_{11}$, $\delta\varepsilon_{22}$, $\delta\hat{\varepsilon}_{12}$, $\delta\kappa_{11}$, $\delta\kappa_{22}$, $\delta\hat{\kappa}_{12}$, $\delta\gamma_{i3}$ and $\delta\gamma_{3i}$ expressed in terms of $\delta v_i$, $\delta w$, $\delta\phi_i$ and $\delta\psi_i$ and their derivatives, eqn (24) implies the seven Euler equilibrium equations,

$$N_{11,1}+N_{12,2} = 0, \qquad N_{12,2}+N_{22,2} = 0, \tag{25}$$

$$M_{11,1}+M_{12,2} = Q_{31}, \qquad M_{12,1}+M_{22,2} = Q_{32}, \tag{26}$$

$$Q_{13,1}+Q_{23,2}+\phi_{1,1}N_{11}+(\phi_{1,2}+\phi_{2,1})N_{12}+\phi_{2,2}N_{22} = 0, \tag{27}$$

$$\begin{aligned} Q_{31}-Q_{13}+(w_{,1}-\phi_1)N_{11}+(w_{,2}-\phi_2)N_{12} &= 0,\\ Q_{32}-Q_{23}+(w_{,1}-\phi_1)N_{12}+(w_{,2}-\phi_2)N_{22} &= 0, \end{aligned} \tag{28}$$

together with the four Euler constitutive relations

$$w_{,i}+\psi_i = \gamma_i, \qquad 2\phi_i = w_{,i}-\psi_i, \tag{29}$$

where (29), with $\gamma_i$ as in (23), complements the constraint constitutive relations (21) and (22). Altogether, we now have in (21), (22) and (25) to (29) a system of 17 equations for the 17 dependent variables $N_{ij} = N_{ji}$, $M_{ij} = M_{ji}$, $Q_{i3}$, $Q_{3i}$, $\phi_i$, $\psi_i$, $v_i$ and $w$.

Given that the transverse shear stress resultants $Q_{i3}$ and $Q_{3i}$ enter into the constitutive equations which are associated with the equilibrium equations (25) to (28) entirely through the combination

$$Q_i = \frac{1}{2}(Q_{i3}+Q_{3i}), \tag{30}$$

it suggests itself to rewrite (26) and (27), with the help of (28) and (25) as follows

$$\begin{aligned} M_{11,1}+M_{12,2} &= Q_1+\frac{1}{2}[(\phi_1-w_{,1})N_{11}+(\phi_2-w_{,2})N_{12}],\\ M_{21,1}+M_{22,2} &= Q_2+\frac{1}{2}[(\phi_1-w_{,1})N_{12}+(\phi_2-w_{,2})N_{22}], \end{aligned} \tag{31}$$

$$Q_{1,1}+Q_{2,2}+\frac{1}{2}[(\phi_{1,1}+w_{,11})N_{11}+(w_{,22}+\phi_{2,2})N_{22}]+(\phi_{1,2}+\phi_{2,1}+2w_{,12})N_{12}] = 0, \tag{32}$$

and to note that as a consequence of (29)

$$w_{,i}-\phi_i = \frac{1}{2}\gamma_i, \qquad w_{,i}+\phi_i = -2\psi_i+\frac{3}{2}\gamma_i. \tag{33}$$

One of the questions which remains to be resolved concerns the relative significance of the $\gamma_i$-terms in the equilibrium equations (31) and (32), relative to the significance of the $Q_i$-terms in the constitutive equations (21) and (22).

## FINITE DEFLECTION EQUATIONS WITH LINEAR CONSTITUTIVE EQUATIONS

Given the general system of constitutive equations (21), (22), and (29) with (23), it is of particular interest to consider cases for which the semicomplementary energy function $V$ contains the strains and stresses by way of second and first degree terms only. Omitting, for simplicity's sake, the effect of first degree terms we can then write, with suitable coefficients $A$, $B$, $C$,

$$V = \frac{1}{2}(A_{11}\gamma_{11}^2 + A_{22}\gamma_{22}^2 + A_{33}\hat{\gamma}_{12}^2) + A_{12}\gamma_{11}\gamma_{22} + A_{13}\gamma_{11}\hat{\gamma}_{12} + A_{23}\gamma_{22}\hat{\gamma}_{12}$$
$$- 2c^2(B_{11}\hat{\tau}_{13}^2 + B_{22}\hat{\tau}_{23}^2 + 2B_{12}\hat{\tau}_{13}\hat{\tau}_{23})$$
$$+ 2c(C_{11}\gamma_{11}\hat{\tau}_{13} + C_{12}\gamma_{11}\hat{\tau}_{23} + C_{21}\gamma_{22}\hat{\tau}_{13} + C_{22}\gamma_{22}\hat{\tau}_{23} + C_{31}\hat{\gamma}_{12}\hat{\tau}_{13} + C_{32}\hat{\gamma}_{12}\hat{\tau}_{23}). \quad (34)$$

In connection with this form of the function $V$ we note in particular that the $B$-terms represent the conventional transverse shear deformation effects, while the $C$-terms represent the effect of laminations which are not parallel to the plane of the undeflected plate. We further note the absence of $\tau_{33}$-terms in (34), which would have to be present for cases without the first of the two approximative assumptions in (15).

For what follows it is of considerable notational convenience to write in (17)

$$\varepsilon_{11} = \varepsilon_1, \quad \varepsilon_{22} = \varepsilon_2, \quad \hat{\varepsilon}_{12} = \varepsilon_3; \quad \kappa_{11} = \kappa_1, \quad \kappa_{22} = \kappa_2, \quad \hat{\kappa}_{12} = \kappa_3, \quad (35)$$

and to write in (21), (22), (25), (31) and (32)

$$N_{11} = N_1, \quad N_{22} = N_2, \quad N_{12} = N_3; \quad M_{11} = M_1, \quad M_{22} = M_2, \quad M_{12} = M_3. \quad (36)$$

Therewith, and with (34), (14) and (30) we obtain from (21) to (23) as a system of *eight* two-dimensional constitutive equations

$$N_i = \varepsilon_j \int A_{ij}\,dx_3 + \kappa_j \int A_{ij}x_3\,dx_3 + Q_j \int C_{ij}\,dx_3, \quad (37)$$

$$M_i = \varepsilon_j \int A_{ij}x_3\,dx_3 + \kappa_j \int A_{ij}x_3^2\,dx_3 + Q_j \int C_{ij}x_3\,dx_3, \quad (38)$$

$$\gamma_i = Q_j \int B_{ij}\,dx_3 - \varepsilon_j \int C_{ji}\,dx_3 - \kappa_j \int C_{ji}x_3\,dx_3. \quad (39)$$

The associated equilibrium equations (25), (31) and (32) become, with the same change of notation

$$N_{1,1} + N_{3,2} = 0, \qquad N_{3,1} + N_{2,2} = 0, \quad (40)$$

$$M_{1,1} + M_{3,2} = Q_1 - \frac{1}{4}(\gamma_1 N_1 + \gamma_2 N_3),$$
$$M_{3,1} + M_{2,2} = Q_2 - \frac{1}{4}(\gamma_1 N_3 + \gamma_2 N_2), \quad (41)$$

$$Q_{1,1}+Q_{2,2} = \left(\psi_{1,1}-\frac{3}{4}\gamma_{1,1}\right)N_1+\left(\psi_{2,2}-\frac{3}{4}\gamma_{2,2}\right)N_2 + \left(\psi_{1,2}-\frac{3}{4}\gamma_{1,2}+\psi_{2,1}-\frac{3}{4}\gamma_{2,1}\right)N_3. \quad (42)$$

With the bending strains $\kappa_i$ given in terms of the $\psi_i$, in accordance with (19), and with $w_{,i}$ and $\phi_i$ obtainable in terms of $\psi_i$ and $\gamma_i$, in accordance with (29), it suggests itself to rewrite the expressions for the membrane strains $\varepsilon_i$ in (29) with $w_{,i}$ and $\phi_i$ expressed in terms of $\psi_i$ and $\gamma_i$. We find after an elementary calculation and upon neglecting second degree terms in $\gamma_i$

$$\varepsilon_1 = v_{1,1}+\frac{1}{2}\psi_1^2-\gamma_1\psi_1, \qquad \varepsilon_2 = v_{2,2}+\frac{1}{2}\psi_2^2-\gamma_2\psi_2,$$
$$\varepsilon_3 = v_{1,2}+v_{2,1}+\psi_1\psi_2-\gamma_2\psi_1-\gamma_1\psi_2. \quad (43)$$

As regards the solution of the system of differential equations as stated we here note as a possible procedure the following. We may use (39) to express the $Q_i$ in terms of the $v_j$, $\psi_j$ and $\gamma_j$. The introduction of these expressions in (37) and (38) then gives the $N_i$ and $M_i$ in terms of the $v_j$, $\psi_j$ and $\gamma_j$. The subsequent introduction of this into the five equilibrium equations in (40) to (42) leaves us with five differential equations for six dependent variables. In order to have a sixth equation we must return to (29) and deduce from this as a further relation involving $\psi_j$ and $\gamma_j$

$$(\psi_1-\gamma_1)_{,2} = (\psi_2-\gamma_2)_{,1}. \quad (44)$$

Given the existence of more practical reduction procedures for the special cases of the problem of finite deflections of non-shear-deformable plates, and for the problem of infinitesimal deflections of shear-deformable plates, it may well be that the above reduction procedure for the general case will turn out to be amenable to further improvements.

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## APPENDIX A: DERIVATION OF THE SEMICOMPLEMENTARY ENERGY FUNCTION $V$

Given a complementary energy function $W = W(\tau_{11}, \hat{\tau}_{12}, \tau_{22}; \hat{\tau}_{13}, \hat{\tau}_{23}, \tau_{33})$ and the set of six constitutive relations $\gamma_{11} = \partial W/\partial\tau_{11}$, $\hat{\gamma}_{12} = \partial W/\partial\hat{\tau}_{12}$, etc., with the $\hat{\gamma}$ and $\hat{\tau}$ as in (2), we deduce a suitable *semi*complementary energy function $V$ upon inverting the subset of *three* constitutive equations

$$\gamma_{11} = \partial W/\partial\tau_{11}, \qquad \hat{\gamma}_{12} = \partial W/\partial\hat{\tau}_{12}, \qquad \gamma_{22} = \partial W/\partial\tau_{22}, \quad (A1)$$

in the form

$$(\tau_{11},\hat{\tau}_{12},\tau_{22}) = (\tau_{11},\hat{\tau}_{12},\tau_{22})(\gamma_{11},\hat{\gamma}_{12},\gamma_{22};\hat{\tau}_{13},\hat{\tau}_{23},\tau_{33}), \quad (A2)$$

and upon then defining $V = V(\gamma_{11}, \hat{\gamma}_{12}, \gamma_{22}; \hat{\tau}_{13}, \hat{\tau}_{23}, \tau_{33})$ as

$$V = \gamma_{11}\tau_{11}(\gamma_{11},\ldots;\hat{\tau}_{13},\ldots)+\ldots-W[\tau_{11}(\ldots),\ldots;\hat{\tau}_{13},\ldots]. \quad (A3)$$

Equation (A3) implies the subset of inverted constitutive equations

$$\tau_{11} = \partial V/\partial\gamma_{11}, \qquad \hat{\tau}_{12} = \partial V/\partial\hat{\gamma}_{12}, \qquad \tau_{22} = \partial V/\partial\gamma_{22}, \quad (A4)$$

and the introduction of (A3) into the variational equation

$$\delta\int[W(\tau_{11},\ldots;\hat{\tau}_{13},\ldots)-\gamma_{11}\tau_{11}-\gamma_{12}\tau_{12}-\ldots-\gamma_{33}\tau_{33}]\,dv=0, \tag{A5}$$

with $\gamma_{ij}$ as in (3) and with independent variations $\delta\tau_{ij}$, $\delta u_k$ and $\delta\chi_m$, where $\delta\alpha_{jk}=e_{ijm}\alpha_{ik}\delta\chi_m$, transforms (A5) into the variational equation (1), with independent variations $\delta u_k$, $\delta\chi_m$, $\delta\tau_{i3}$, $\delta\tau_{3i}$ and $\delta(\tau_{12}-\tau_{21})$ [2].

For the case of a complementary energy function

$$W=W_0(\hat{\tau}_{13},\hat{\tau}_{23},\tau_{33})+W_1(\hat{\tau}_{13},\ldots;\tau_{11},\ldots)+W_2(\tau_{11},\hat{\tau}_{12},\tau_{22}), \tag{A6}$$

where $W_0$ and $W_2$ are homogeneous of the second degree and $W_1$ is homogeneous of the first degree, the three constitutive equations (A1) are a system of three *linear* equations for $\tau_{11}$, $\hat{\tau}_{12}$, $\tau_{22}$, of the form

$$\frac{\partial W_2}{\partial\tau_{11}}=\gamma_{11}-\frac{\partial W_1}{\partial\tau_{11}},\qquad \frac{\partial W_2}{\partial\hat{\tau}_{12}}=\hat{\gamma}_{12}-\frac{\partial W_1}{\partial\hat{\tau}_{12}},\qquad \frac{\partial W_2}{\partial\tau_{22}}=\gamma_{22}-\frac{\partial W_1}{\partial\tau_{22}}, \tag{A7}$$

with solution functions which depend linearly on $\gamma_{11}$, $\hat{\gamma}_{12}$, $\gamma_{22}$ and $\hat{\tau}_{13}$, $\hat{\tau}_{23}$, $\tau_{33}$.

The introduction of these solution functions into the defining relation (A3), in the form

$$V=\left(\frac{\partial W_1}{\partial\tau_{11}}+\frac{\partial W_2}{\partial\tau_{11}}\right)\tau_{11}+\ldots-W_0-W_1-W_2, \tag{A8}$$

and an observation of the homogeneity consequences

$$\tau_{11}\,\partial W_1/\partial\tau_{11}+\ldots=W_1,\qquad \tau_{11}\,\partial W_2/\partial\tau_{11}+\ldots=2W_2, \tag{A9}$$

gives as expression for $V$

$$V=W_2[\tau_{11}(\gamma_{11},\ldots;\hat{\tau}_{13},\ldots),\hat{\tau}_{12}(\ldots),\tau_{22}(\ldots)]-W_0(\hat{\tau}_{13},\ldots). \tag{A10}$$

In order to see the nature of the coefficients $A$, $B$, $C$ in (34) in terms of the set of analogous coefficients in (A6) we write

$$W_2=\frac{1}{2}(a_{11}\tau_{11}^2+a_{22}\tau_{22}^2+a_{33}\hat{\tau}_{12}^2)+a_{12}\tau_{11}\tau_{22}+a_{13}\tau_{11}\hat{\tau}_{12}+\ldots, \tag{A11}$$

$$W_1=c_{11}\tau_{11}\hat{\tau}_{13}+c_{12}\tau_{11}\hat{\tau}_{23}+c_{13}\tau_{11}\tau_{33}+\ldots+c_{33}\hat{\tau}_{12}\tau_{33}, \tag{A12}$$

$$W_0=\frac{1}{2}(b_{11}\hat{\tau}_{13}^2+b_{22}\hat{\tau}_{23}^2+b_{33}\tau_{33}^2)+b_{12}\hat{\tau}_{13}\hat{\tau}_{23}+b_{13}\hat{\tau}_{13}\tau_{33}+\ldots. \tag{A13}$$

Equations (A7) now become, with $a_{ij}=a_{ji}$,

$$\begin{aligned}a_{11}\tau_{11}+a_{12}\tau_{22}+a_{13}\hat{\tau}_{12}&=\gamma_{11}-c_{11}\hat{\tau}_{13}-c_{12}\hat{\tau}_{23}-c_{13}\tau_{33},\\ a_{21}\tau_{11}+\tau_{22}\tau_{22}+a_{23}\hat{\tau}_{12}&=\gamma_{22}-c_{21}\hat{\tau}_{13}-c_{22}\hat{\tau}_{23}-c_{23}\tau_{33},\\ a_{31}\tau_{11}+a_{32}\tau_{22}+a_{33}\hat{\tau}_{12}&=\hat{\gamma}_{12}-c_{31}\hat{\tau}_{13}-c_{32}\hat{\tau}_{23}-c_{33}\tau_{33}.\end{aligned} \tag{A14}$$

The introduction of the solutions $\tau_{11}$, $\tau_{22}$, $\hat{\tau}_{12}$ of the system (A14) into (A11) and the subsequent introduction of the resulting expression, together with $W_0$ from (A13), into (A10) gives $V$ as the desired quadratic form in $\gamma_{11}$, $\gamma_{22}$, $\hat{\gamma}_{12}$ and $\hat{\tau}_{13}$, $\hat{\tau}_{23}$, $\tau_{33}$. The case which is considered in the body of the text represents a slight practical simplification of the above, which results upon stipulating $c_{i3}=0$ in (A12) and $b_{i3}=0$ in (A13).

## APPENDIX B: A PARAMETRIC REPRESENTATION FOR ROTATIONS

Given the set of six orthonormality conditions $\alpha_{ik}\alpha_{jk}=\delta_{ik}$ for the nine coefficients in the triad relations $\mathbf{t}_j=\alpha_{jk}\mathbf{e}_k$ it is to be expected that the $\alpha_{ij}$ can be expressed in terms of three unrestricted parameters. Of the various ways in which this can be done we here describe one, which is unsymmetrical in a fashion appropriate to its use for the analysis of *plates*.

We begin by introducing an angular rotation measure $\beta_3$ in the plane of the plate and a system of planar rotated unit vectors $\hat{\mathbf{e}}_i$, in the form

$$\hat{\mathbf{e}}_1=\mathbf{e}_1c_3+\mathbf{e}_2s_3,\qquad \hat{\mathbf{e}}_2=\mathbf{e}_2c_3-\mathbf{e}_1s_3, \tag{B1}$$

where $c_i\equiv\cos\beta_i$ and $s_i\equiv\sin\beta_i$.

Subsequent to this we introduce two angular measures $\beta_1$, $\beta_2$ for rotation out of this plane by writing

$$h\mathbf{t}_1=\hat{\mathbf{e}}_1c_1+\mathbf{e}_3s_1-\hat{\mathbf{e}}_2g,\qquad h\mathbf{t}_2=\hat{\mathbf{e}}_2c_2+\mathbf{e}_3s_2-\hat{\mathbf{e}}_1g, \tag{B2}$$

where $h=(1+g^2)^{1/2}$ and $g=s_1s_2/(c_1+c_2)$, so as to have $\mathbf{t}_i\cdot\mathbf{t}_j=\delta_{ij}$.

Having $\mathbf{t}_1$ and $\mathbf{t}_2$ we write $\mathbf{t}_3 = \mathbf{t}_1 \times \mathbf{t}_2$. With (B2) this makes

$$h^2\mathbf{t}_3 = \mathbf{e}_3(c_1c_2 - g^2) - \hat{\mathbf{e}}_1(s_1c_2 + s_2g) - \hat{\mathbf{e}}_2(s_2c_1 + s_1g). \tag{B3}$$

The introduction of (B1) into (B2) and (B3) leads to the following expressions for the coefficients $\alpha_{ij}$.

$$\begin{aligned} h\alpha_{11} &= c_3c_1 + s_3g, \qquad h\alpha_{12} = s_3c_1 - c_3g, \qquad h\alpha_{13} = s_1, \\ h\alpha_{21} &= -s_3c_2 + c_3g, \qquad h\alpha_{22} = c_3c_2 - s_3g, \qquad h\alpha_{23} = s_2, \\ h^2\alpha_{31} &= s_3s_2c_1 - c_3s_1c_2 + s_3s_1g - c_3s_2g, \\ h^2\alpha_{32} &= -s_3s_1c_2 - c_3s_2c_1 - s_3s_2g - c_3s_1g, \\ h^2\alpha_{33} &= c_1c_2 - g^2. \end{aligned} \tag{B4}$$

Retention of no more than second degree terms in $\beta_1$ and $\beta_2$ and first degree terms in $\beta_3$ in (B4) results in the approximate relations in (7).